Enriching Number Knowle

Exploring number systems of other cultures helps students deepen mental computation fluency, knowledge of place value, and equivalent representations for numbers.

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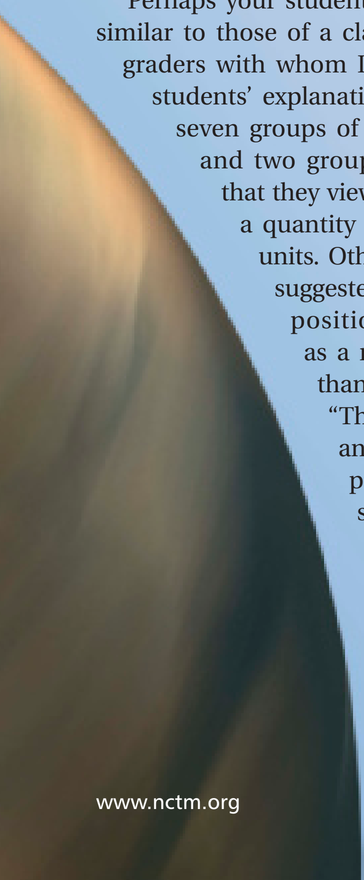
By Nancy K. Mack

If you asked your students to explain the meaning of a number, such as 237, what would they say? If you asked them to represent the number in equivalent ways, how would they do so, and what would their representations suggest about their understanding of numbers?

Perhaps your students' responses would be similar to those of a class of fourth and fifth graders with whom I worked. Some of the students' explanations—such as, "There's seven groups of 1, three groups of 10, and two groups of 100"—suggested that they viewed the number 237 as a quantity composed of different units. Other students' responses suggested that they viewed the position of each digit only as a name or a label rather than as a quantity or a unit: "Three is in the tens place, and seven is in the [units] place." Additionally, some students readily decomposed numbers and represented them in a variety of ways that demonstrated flexibility of thought and an understanding of relationships between different units, such as expressing 237 as "twenty-three groups of ten and seven ones" or "twenty-two units of ten and seventeen units of one." However, other students were able to represent the number only in a few straightforward ways, such as 200 + 30 + 7. Student responses showed that their knowledge of numbers varied in important ways. *Principles and Standards for School Mathematics* suggests that all students should conceptually understand the place-value structure of our number system, see relationships between numbers, be able to recognize and generate equivalent representations for numbers, and demonstrate mental computation fluency (NCTM 2000). Such knowledge requires thinking flexibly about numbers and viewing numbers as quantities composed of different-size units or groups.

I wanted to help the fourth- and fifth-grade students to whom I gave a weekly enrichment lesson develop the rich knowledge of numbers described in *Principles and Standards*. In particular, one of my goals was that all students would deepen their knowledge of place value and understand how units are involved in the structure of our base-ten number system. My other goals were that students would become more flexible and creative in the ways they thought about and represented numbers, as well as become more fluent in mental computations. One challenge was how to achieve these goals when the students' knowledge of numbers varied greatly. I decided it would be helpful for students to explore numbers from a different perspective, one that was new and unfamiliar to them. Doing so had the potential to minimize important differences in students' prior knowledge of numbers and to help them focus on the structure underlying a number system. Thus, for seven months, I engaged my classes in exploring number systems of different cultures once a week



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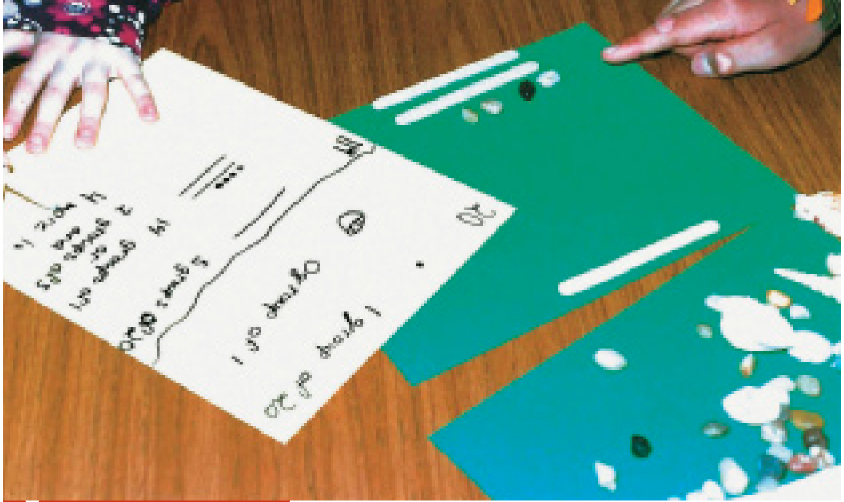


For several numbers, we used representations of the Mayan symbols and explained the numerical representations by referring to the unit structure of the number system.

in one-hour segments. These explorations helped students grow in their knowledge of place-value ideas, equivalent representations for numbers, and mental computational fluency.

Number systems

I designed our explorations to focus on different cultures and different types of number systems. We explored the Hindu-Arabic, Mayan, and Babylonian place-value systems. We also explored Roman numerals and the Egyptian system, which are both additive number systems. In an *additive number system*, numbers are represented by writing symbols of particular values a certain number of times. For example, Olivia represented 12 with Egyptian numbers by writing the symbol for 1 two times and writing the symbol for 10 once. For 302 she wrote the symbol for 100 three times and the symbol for 1 two times (see **fig. 1**).

The Hindu-Arabic system

We first focused on exploring the unit structure of our Hindu-Arabic base-ten number system. I thought such an exploration would deepen students' understanding of grouping by tens to form different units as well as later aid them when comparing the structure of various number systems. To initially generate student interest, I read *On Beyond a Million: An Amazing Math Journey* (Schwartz and Meisel 1999), which presents a story of counting by units that are powers of ten. The book also imparts interesting realistic facts about large numbers. As I read, students began enthusiastically telling the story of the unit development by focusing on groups of ten. They eagerly offered such comments as, "I know what happens next! It's going to be one hundred thousand because ten groups of ten thousand makes a hundred thousand."

To further focus students on the unit structure of the Hindu-Arabic number system, I asked them to determine the number of jumbo-size colored paper clips in a twelve-inch-high jar. Each student received a large number of paper clips to count. Students agreed, "We should count them by making groups of ten," which they proceeded to do. We used the groups of ten to create different units and determine the total quantity.

We created units of 1 paper clip, 10 paper clips, 100 paper clips, and 1000 paper clips. We continued creating various units until we had used all the paper clips. Students referred to the units to determine the total number of paper clips:

> There's two groups of 1000 paper clips, four groups of 100 paper clips, three groups of 10 paper clips, and 7 extra paper clips; so there's 2407 paper clips. That's a lot of paper clips!

Following the Paper-clip activity, I told students that not all people and cultures group by tens when working with numbers, nor does everyone use the same symbols we use to represent numbers. Eager to know more, they asked, "What do they do? Can we find out what they do?" Thus began our exploration of different number systems.

The Mayan system

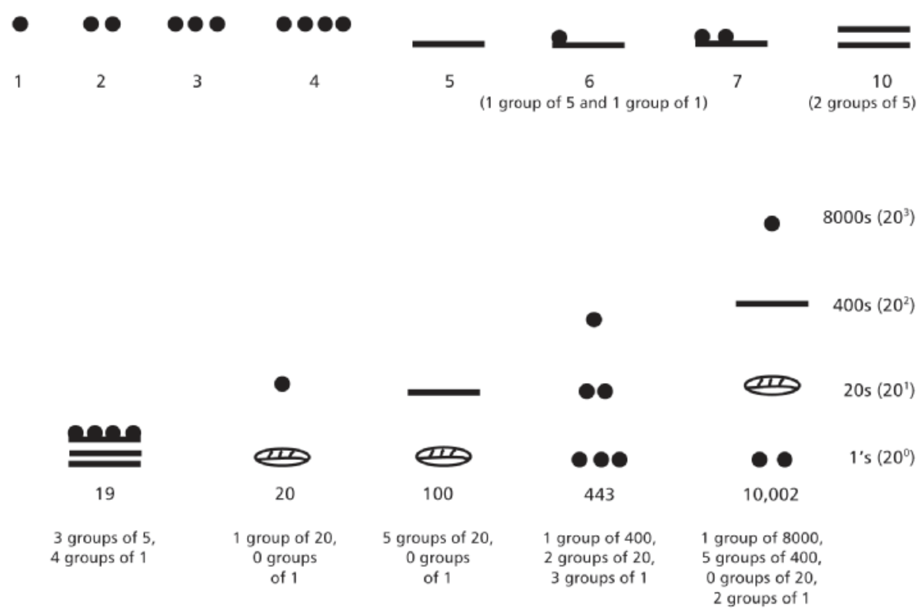
Referred to more inclusively as the *Mesoamerican number system*, the Mayan number system (see **fig. 2**) is an important part of Hispanic culture (Krause 2000; Ortiz-Franco 2006; Zaslavsky 1996, 2001). I thought exploring this system would deepen students' understanding of using consistent groupings to form units, be challenging but accessible to all students, and represent a culture

FIGURE 1

Olivia learned to represent 12 and 302 with Egyptian numbers.

FIGURE 2

The Mayan number system is a base-twenty place-value number system that uses vertical positional notation.



for which some of them had a personal connection or interest.

I introduced the Mayan number system by asking students to locate on a world map the geographic region known as *Middle America* (southern Mexico to northern Central America), where the Mayan number system has been widely used for more than 2000 years. We also discussed characteristics of the geographic region and the people who live there, as well as the types of math problems the people might have solved in the past and might solve today. Following our discussion, I remarked that the Mayan number system uses only three symbols to represent any whole number, zero to a desired number. Students curiously asked, "What symbols do they use?"

Ancient Mayans laid out twigs, pebbles, and shells on the ground to perform everyday marketplace calculations. The Mayan symbols of dots, bars, and shells were likely derived from calculation activities with these materials (Zaslavsky 1996, 2001). Thus, I introduced the Mayan symbols by giving students bags containing sticks, stones, and shells. (We used craft sticks instead of tree twigs because twigs can get moldy when kept in plastic storage bags.)

We started exploring the Mayan number system by determining the value of the three symbols as we represented various numbers with the sticks, stones, and shells. After students became somewhat proficient in representing and writing numbers, we created and solved realistic contextual problems similar to what the Mayans might have solved, such as those involving agricultural situations. We also engaged in other activities that focused on representing numbers in equivalent ways and using mental computations to solve problems. Throughout our exploration, we used strategies the Mayans used to solve problems, and we compared numerical representations in the Mayan number system with those in the Hindu-Arabic number system.

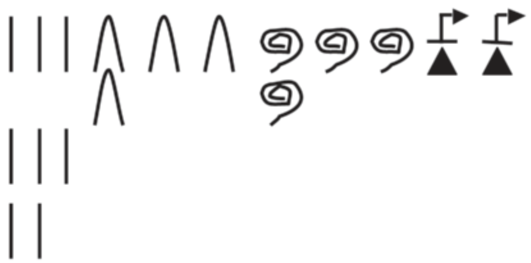
Students found the Mayan number system to be an enjoyable challenge. Kenya said, "This is hard, but it's fun," and Bryce commented, "I like it because it's challenging." Thomas found the number system and culture so interesting that he investigated the Mayans on his own, each week eagerly sharing with the class what he had learned.

Roman, Babylonian, and two additive systems

After the Mayan number system, we explored two additive number systems to help students

FIGURE 3

Students accurately described the author's answer in the Egyptian system as the Hindu-Arabic number 2448.



For each number system, students engaged in a problem-solving activity focused on groups or units and their numerical symbols.

gain a different perspective on the unit structure of the Hindu-Arabic number system. The Egyptian number system uses different symbols to represent numbers that are powers of ten. We then explored Roman numerals, concluding our exploration by returning to a place-value number system, the base-sixty system developed by the Babylonians. We investigated each of these number systems in a manner similar to how we had explored the Mayan number system, which included discussing the culture that developed the number system, representing numbers, solving both realistic contextual and computational problems, and comparing characteristics of the different number systems.

To introduce each number system, I engaged students in a problem-solving activity that focused their attention on groups or units and the numerical symbols used in a particular system. For example, I introduced the Egyptian number system by asking students to determine the number of dots on a 30 in. × 96 in. piece of wrapping paper, which they did by finding various arrays and determining the number of array units in the large piece of paper. After they found their answers, I showed my answer (see **fig. 3**) and challenged them to determine the meaning of the symbols and what culture might have used the symbols.

Children's literature played an important role in our explorations. We began each hour by reading a portion of a book. The books helped students learn more about the culture that developed a particular number system and encouraged them to think about important mathematical ideas we would explore that day. The children looked forward to this activity and often eagerly asked, "What are we going to read today?"

During our exploration of Egyptian numbers, we read *Senefer: A Young Genius in Old Egypt* (Lumpkins and Nickens 1991), which tells the story of a peasant boy in ancient Egypt who was invited to go to school because of his ability to solve mathematical problems without being taught. The book helped students understand strategies that ancient Egyptians used to add and subtract numbers, which they found highly interesting. As Sadiel said, "I think it's really cool how Senefer solved problems."

We read the chapter book *Number Stories of Long Ago* (Smith 2001) during our exploration of Roman numerals and the Babylonian number system. This book explains why numbers were developed by various cultures in different parts of the world, how different cultures wrote and operated on numbers, and how number symbols and number systems in different parts of the world contributed to one another as well as evolved over time. Several students commented, "This book was my favorite." Eric-John claimed, "All the books we read were sensational!"

Students enjoyed learning about the number systems developed by different cultures. During our exploration of the Egyptian number system, students frequently said, "This one is easy. This is fun!" Regarding Roman numerals, several students commented, "This one is my favorite." They enthusiastically shared examples of Roman numerals they found in their environment, such as the page numbers and chapter titles of books they were reading, numbers on clock faces, and numbers in family members' names. Alexis began writing the date in Roman numerals on all her papers. Throughout our explorations, Thomas continued to learn more about various cultures on his own and enthusiastically share his new knowledge with the class. At the end of our explorations, Eric-John proudly presented me with a chart he had created on his own initiative to confirm what he had learned. His chart showed key numbers in the different systems we had explored and how to represent these numbers in the systems (see **fig. 4**).

Much information about various cultures and their number systems can readily be found in books and articles on multicultural mathematics (see the references section of this article). Additionally, Internet searches for different number systems can locate photographs showing the numbers on real-life objects used by the culture, such as coins, buildings, and drawings.

Place value

My students' understanding of the unit structure of the Hindu-Arabic number system and other place-value number systems grew deeper as they learned to represent numbers in different systems. In particular, students grasped the idea that units or groups are formed in consistent ways, such as grouping by tens, and that different units have different values. Much of this growth was linked to the exploration of the Mayan number system.

When students first examined the sticks, stones, and shells used by the Mayans, they logically but erroneously assigned values to the materials by referring to units in the Hindu-Arabic number system. Ivory explained:

> The pebbles (stones) are the ones, because they are really easy to find. The sticks are tens, because they are harder to find; and the shells are worth the most, like a hundred, because they are hard to find.

I commented, "A stone does represent one of something."

We then used the stones to represent the numbers 1–4. The students suggested five stones to represent the number 5.

I explained, "Yes, that is one way to show five, but the Mayans show five in a different way. Their number system uses subgroups of five. They use a stick to show five."

The students proceeded to accurately represent the numbers 5–19 by decomposing each number into groups of fives and ones. Parker explained his representation for the number 17: "There's three groups of five, that's fifteen, and two more is seventeen."

When I asked how the Mayans would represent the number 20, students unanimously proposed, "Use four sticks, because it's four groups of five." Following this, I commented,

> The Mayans have a different way to represent twenty. Their number system is based on making groups of twenty just like the Hindu-Arabic number system involves making groups of ten. How do you think we can show there is one group of twenty and no extras?

FIGURE 4

Eric-John represented key numbers in all the different number systems the class had explored.

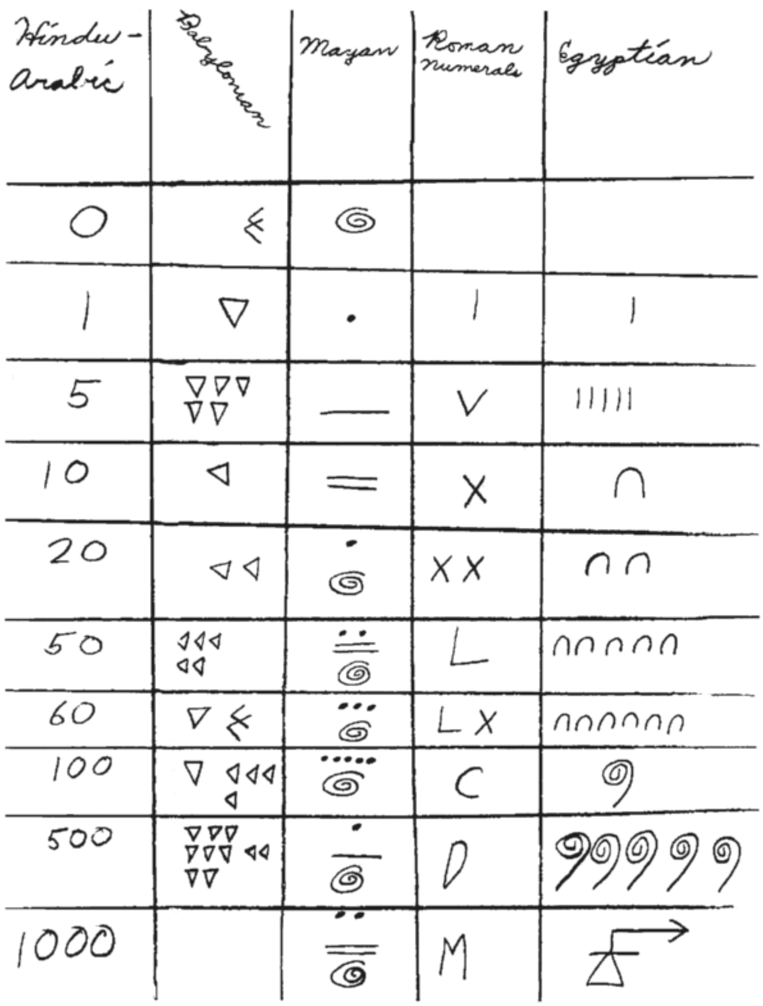
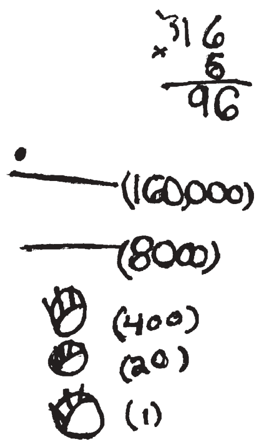


FIGURE 5

Gabby represented one million with Mayan numbers.



Initially, everyone was puzzled. They tried representing the number 20 in a variety of ways, but their representations did not show the unit structure of the Mayan number system. After some time, I asked them to think about how the number 10 is represented in the Hindu-Arabic number system and what each digit in the number means in terms of units or groups. A long pause followed. Suddenly, Eric-John exclaimed,

> I know! I get it! The one means there's one group of ten, and the zero means there's zero groups of one. To show twenty in the Mayan number system, we need one group of twenty. I think we can use the pebble to show one group of twenty. We also need to show zero groups of one. How can we show zero? Can we use the shell to show zero? We haven't used it yet.

I responded, "A shell does represent zero of something in the Mayan number system."

We then represented the number 20 by using one stone, one shell, and positional notation to show one group of 20 and zero groups of 1 (see the photograph on p. 102). Some students quickly grasped the different units involved. Other students had to wrestle with representations for the numbers 21–24 before they comprehended the unit structure. These students initially represented twenty-one by adding one stone next to the shell in the representation for twenty, or showing twenty plus one. Students who understood the different units helped those who did not yet understand. Soon all the students were talking about the number of groups of twenty, the number of groups of five, and the number of groups of one and constructing appropriate representations for numbers.

Many students readily extended the idea of grouping by twenty to determine units of greater value in the Mayan number system. They freely referred to the units to justify numerical representations they created for large numbers. For example, Gabby represented one million with Mayan numbers by showing that the quantity was composed of six groups of 160,000, five groups of 8,000, and zero groups of units of lesser value (see **fig. 5**).

Students continued to focus on groups or units on their own as they worked with quantities represented in other number systems, regardless of whether it was a place-value or additive-number system. For example, when I had written my Egyptian number (see **fig. 3**) to answer how many dots were on the large piece of wrapping paper, students had referred to their own answers and had quickly explained, "That's two groups of one thousand, four groups of one hundred, four groups of ten, and eight groups of one," which accurately describes my number. The students' focus on groups and units helped them refer to numbers in the Hindu-Arabic number system in terms of groups or units rather than by referring to place value as names or labels. This focus also helped the children realize that the number 0 is an important and necessary number in a place-value number system. When Egyptian numbers were introduced, Cheyla asked, "How do they write zero?" She and other students expressed concern when they learned that a number for zero did not exist in this number system. Several students asked, "How would we know if a • meant one, twenty, or some other number in the Mayan number system if there was nothing to show zero?" They further commented, "I'm glad we [who use the Hindu-Arabic number system] have zero, so we know what a number means."

Equivalent representations

During our explorations of different number systems, these fourth and fifth graders became more creative in representing numbers in equivalent ways. As they solved problems, they also became more aware of when and how to rename numbers in equivalent ways. This growth was largely related to activities that explicitly focused on equivalent representations for numbers.

During our exploration of the Mayan number system, we played a game the students named Mayan Go Fish, involving matching equivalent representations for numbers. For example, a card showing the representation for 9×6 could be matched with a card showing the Mayan representation of the number 54. As students played the game, they began explaining numerical representations in a variety of equivalent ways. For example, some students explained the representation for 54 as, “It’s twenty plus twenty, that’s forty; plus five, that’s forty five; plus five more, that’s fifty; plus four more is fifty four.”

Other students explained the representation in this way: “It’s two groups of twenty. Two times twenty is forty. There’s two groups of five. Two times five is ten. There’s four more, so forty plus ten plus four is fifty-four.”

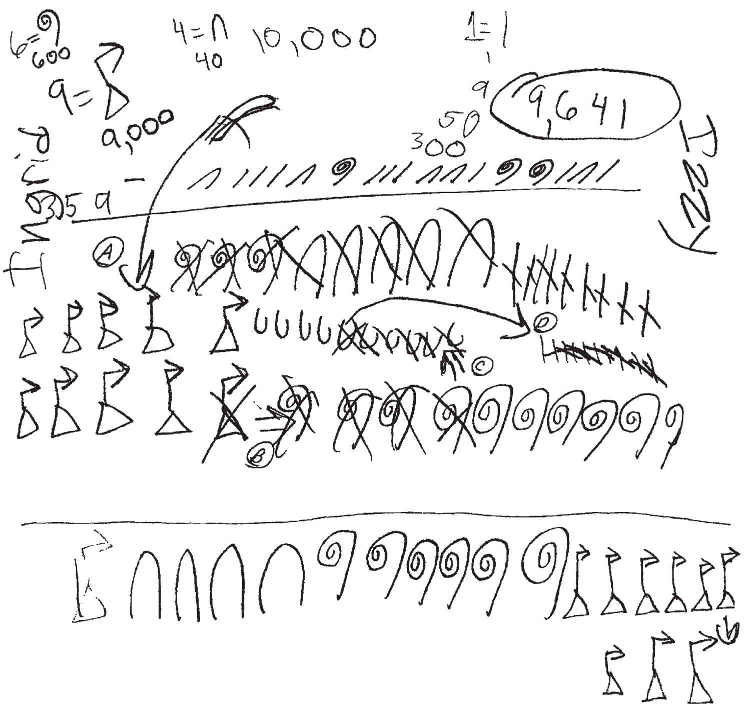
Playing games like Mayan Go Fish contributed to students’ fluency in decomposing numbers and computing mentally.

Students enjoyed the challenge of determining and matching equivalent representations. They often commented, “This game is my favorite,” and asked, “Can we play Mayan Go Fish today?” (For directions and number cards for Mayan Go Fish, see the **appendix online**.)

Students drew on their knowledge of equivalent representations to rename numbers in appropriate ways as they used strategies the cultures used to solve problems. For example, Isabel and Ingrid encountered the problem 10,000 – 359, represented with Egyptian numerals. They first renamed 10,000 as 9000 + 900 + 90 + 10 so they could try a matching strategy used by ancient Egyptians to solve the problem. Their strategy involved matching three of the hundreds, five of the tens, and nine of the units from both numbers to determine the answer (see **fig. 6**). Similarly, when solving contextual problems involving dates on coins and buildings that

FIGURE 6

Isabel and Ingrid used equivalent representations when solving 10,000 – 359.



were represented with Roman numerals, Ivory renamed the number C (100) as LXXXXVIIIII (50 + 10 + 10 + 10 + 10 + 5 + 1 + 1 + 1 + 1 + 1). Alexis renamed the number CCC (300) as LLLLLL (50 + 50 + 50 + 50 + 50 + 50), and Thomas renamed the number X (10) as VV (5 + 5).

The growth in students' knowledge of equivalent representations extended to their regular mathematics work. Their explanations for their strategies became more conceptually rich as they explained in depth why they renamed numbers in equivalent ways and justified the equivalence of their representations when solving problems with numbers represented in the Hindu-Arabic number system.

Mental computational fluency

Students also became more fluent in mental computations as they solved problems and represented numbers in different number systems. During our early explorations, several students needed time or assistance to mentally determine the value of Mayan numbers. Additionally, we needed to first cooperatively play Mayan Go Fish to make the game accessible to all.

By the end of our explorations, everyone was able to independently perform mental computations with great fluency. The children readily combined groups to determine numbers represented in different systems. They also mentally decomposed numbers in various ways to generate equivalent representations. For example, on the last day, we played a variation of bingo that involved numbers represented in all the number systems we had explored. During the game, Skyler quickly determined the value of the Mayan number for 92. He explained, "I figured it out in my head. Four groups of twenty is eighty; plus two groups of five makes ninety; plus two more is ninety-two." Students eagerly offered additional equivalent representations for 92, which they mentally determined by thinking of other number systems we had explored:

- **One** group of fifty plus four groups of ten and two ones
- **One** hundred minus ten, plus two (Roman numerals)
- **One** group of sixty plus three groups of ten and two ones (Babylonian number system)
- **Nine** groups of ten and two ones (Egyptian number system)

Conclusion

Exploring number systems of different cultures can be an enjoyable learning experience that enriches students' knowledge of numbers and number systems in important ways. The explorations can deepen students' understanding of place-value ideas, challenge students to think flexibly about numbers, and help students appreciate numbers and the contributions different cultures have made to the development of number systems. Such explorations can effectively occur in small increments over an extended period of time. Additionally, they can be designed to serve as an alternate way to review concepts and skills that students are learning. Such was the Mayan Go Fish game and the various contextual and computational problems that students solved during our explorations.

As your students explore number systems of different cultures, perhaps you will not only see their knowledge of numbers deepen but also hear them offer comments similar to those I heard: "I thought this year was a great experi-

ence learning all about how different people use numbers," reported Lisa.

Sarah agreed: "This is fun. Can we learn more?"

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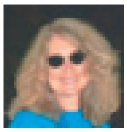
After Skyler explained his mental computation and the Mayan representation of the number, other students eagerly offered additional equivalent representations from other number systems.



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Go to **www.nctm.org/tcm** to access the **appendix**, which accompanies the online version of this article.